Dynamical Behavior of a System of Interacting Vortices in a Confined Bose-Einstein Condensate

N. Kyriakopoulos[†], V. Koukouloyannis[†], Ch. Skokos[†], P. Kevrekidis[‡], G. Voyatzis[†]
[†]Dept. of Physics, Aristotle University of Thessaloniki
[‡]Dept of Mathematics and Statistics, University of Massachussets Amherst

Problem

In a system of interacting vortices in a pancake-shaped, confined Bose-Einstein condensate with a harmonic trap (with frequency $\Omega \to 0$), the motion of each vortex, sufficiently far away from the others and the condesnsate's boundary, is a combination of the precession around the trap and the interaction with other vortices, sufficiently far away from the vortex in question.

The dimensionless equations of motion of our system are

$$\dot{x}_i = -S_i \frac{y_i}{1-r_i^2} - c \sum_{j=1, j\neq 1}^{N} S_j \frac{y_i - y_j}{r_{ij}^2}, \quad \dot{y}_i = S_i \frac{x_i}{1-r_i^2} + c \sum_{j=1, j\neq 1}^{N} S_j \frac{x_i - x_j}{r_{ij}^2}, \text{ with } \quad r_i = \sqrt{x_i^2 + y_i^2} \text{ and } \quad r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

where c is a parameter which depends on the physical parameters of the system and in our case is $c = 0.1$. These equations can also be derived from the Hamiltonian $H = \frac{1}{2}\sum_{k=1}^{N} \ln(1-r_k^2) + \frac{c}{2}\sum_{k=1}^{N}\sum_{k\neq j} S_k S_j \ln(r_{kj}^2)$ through the canonical equations $\dot{x}_i = S_i \frac{\partial H}{\partial y_i}$ and $\dot{y}_i = -S_i \frac{\partial H}{\partial x_i}$.

For $N = 3$ and $S_1 = S_3 = +1$, $S_2 = -1$, we define the generalized positions to be $\mathbf{q} = (x_1, y_2, x_3)$ and the generalized momenta to be $\mathbf{p} = (y_1, x_2, y_3)$ and obtain the standard equations of motion. By applying canonical transformations, the Hamiltonian becomes

$$H = \frac{1}{2}\left[\ln(1-2J_1) + \ln(1-2J_2) + \ln(1-2(L-J_1+J_2))\right] +$$
$$\frac{c}{2}\left[\ln(4J_2 - 2J_1 + 2L - 4\sqrt{J_2}\sqrt{L-J_1+J_2}\sin(\phi_2)) - \ln(2L + 2J_2 - 4\sqrt{J_1}\sqrt{L-J_1+J_2}\cos(\phi_1)) + \ln(2J_1 + 2J_2 - 2\sqrt{J_1}\sqrt{J_2}\sin(\phi_1+\phi_2))\right]$$

For this Hamiltonian, the variable L is an integral of motion -the corresponding generalized position does not appear explicitly in H- and thus the system can be considered as a 2-degrees of freedom system with L as a parameter.

Method

We considered a 300×300 grid of initial conditions $(\phi_1 \times J_1)$ for fixed values of h and L. For each initial condition we calculated the value of SALI for a maximum value of time $t_{max} = 3000$ and created a heatmap in which colors correspond to the value of the time needed for it to become smaller than 10^{-12}.

Finally, we eliminated areas of collision, i.e. initial conditions for which the vortices are too close to each other and have no physical meaning (since they have a finite diameter).

The Smaller ALignment Index (SALI)

For a $2N$-dimensional phase space of a Hamiltonian flow and an orbit with initial condition $P(0) = (x_i(0))_{i=1}^{2N}$, the evolution in time of a deviation vector $v(0) = (\delta x_i)_{i=1}^{2N}$ is defined by the variational equations.

Following the evolution of two different deviation vectors, $v_1(0)$ and $v_2(0)$, SALI is defined as

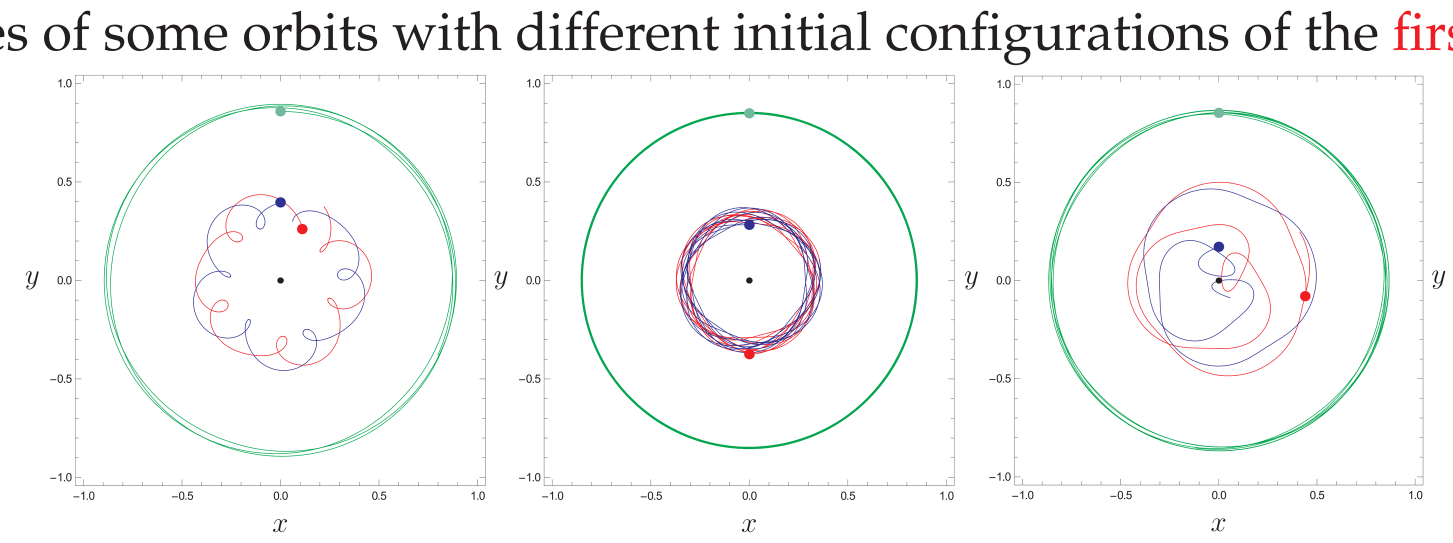
$$\text{SALI}(t) = min\left\{\|\hat{v_1}(t) + \hat{v_2}(t)\|, \|\hat{v_1}(t) - \hat{v_2}(t)\|\right\}$$

with $\hat{v}_i(t) = \frac{v_i(t)}{\|v_i(t)\|}$.

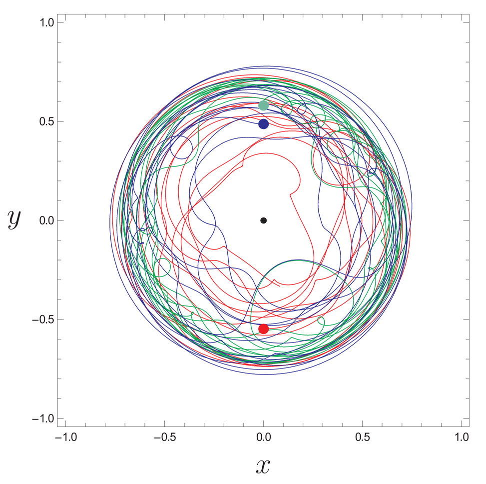
When the orbit is chaotic the two deviation vectors tend to become collinear, so SALI$\to 0$. When the orbit is regular, the vectors become tangent to the invariant torus containing the orbit and are generally non-parallel so SALI$\to$ const. $\neq 0$.

Results

Examples of some orbits with different initial configurations of the first, second and third particle.

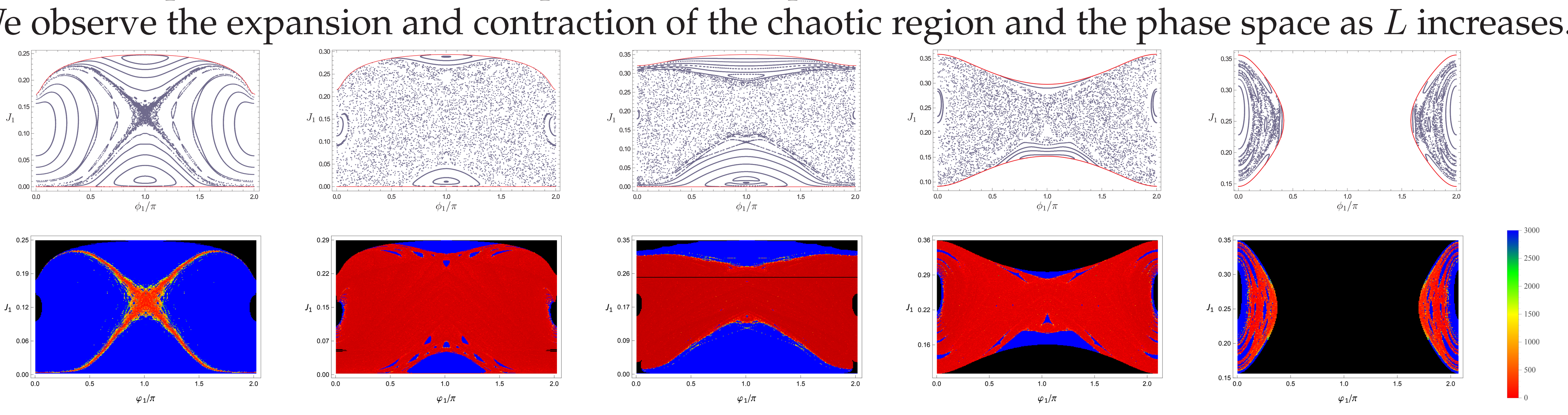


Three qualitatively different types of regular orbits: (a) Two of the vortices rotate around each other, (b) All three vortices revolve around the center of the BEC, (c) One of the vortices passes through the center



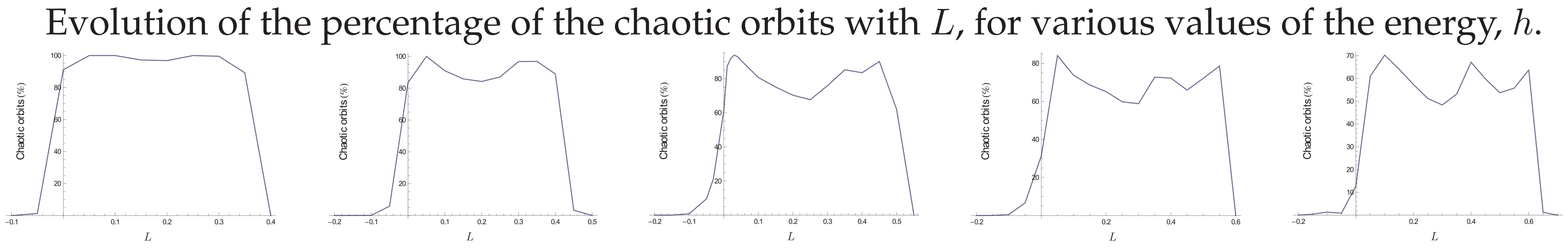
A chaotic orbit: the orbits intertwine with each other

Comparison of Poincaré maps and Heatmaps for $h = -0.75$ and various values of L.
We observe the expansion and contraction of the chaotic region and the phase space as L increases.



L=-0.05 L=0.05 L=0.25 L=0.45 L=0.50

Evolution of the percentage of the chaotic orbits with L, for various values of the energy, h.



h=-0.5 h=-0.6 h=-0.75 h=-0.9 h=-1.1

Conclusions - Future

This system presents similar behaviour for every value of the energy h inside the range we have considered. There is a value of L for which a chaotic region appears and as L grows larger it blows up, contracts for some values and then returns to about the previous maximum area before starting to disappear. The exact range of L for which the whole process takes place is not constant, but is determined by h.

In addition to being able to get quantitative estimates of the area of the chaotic region, the use of SALI can be helpful in systems of more than three interacting vortices, where the visual representation of the results is harder, or even impossible. In those systems we should also be able to determine the size of the stable region around the steady states of the system.

References

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